

A QUESTION OF SÁRKÖZY AND SÓS ON REPRESENTATION FUNCTIONS

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ABSTRACT. For $m \geq 1$, let $0 < b_0 < b_1 < \cdots < b_m$ and $e_0, e_1, \dots, e_m > 0$ be fixed positive integers. Assume there exists a prime p and an integer $t > 0$ such that $p^t \mid b_0$, but $p^t \nmid b_i$ for $1 \leq i \leq m$. Then, we prove that there is no infinite subset \mathcal{A} of positive integers, such that the number of solutions of the following equation

$$n = b_0(a_{0,1} + \cdots + a_{0,e_0}) + \cdots + b_m(a_{m,1} + \cdots + a_{m,e_m}), \quad a_{i,j} \in \mathcal{A}$$

is constant for n large enough. This result generalizes the recent result of Cilleruelo and Rué for the bilinear case, and answers a question posed by Sárközy and Sós.

1. INTRODUCTION

Given an infinite subset \mathcal{A} of positive integers \mathbb{N} , the representation function $r(n, \mathcal{A})$ is defined as

$$r(n, \mathcal{A}) = \#\{(a, a') \mid n = a + a', \quad a, a' \in \mathcal{A}\}.$$

This function was initially studied by Erdős and Turán [4]. In that paper, they made the following important conjecture.

Conjecture: (Erdős and Turán): If $\mathcal{A} \subseteq \mathbb{N}$ and $r(n, \mathcal{A}) > 0$ for $n > n_0$ (i.e., \mathcal{A} is an asymptotic basis of order 2), then $r(n, \mathcal{A})$ cannot be bounded.

As an evidence, Erdős and Turán [4] found, by means of analytic arguments, that $r(n, \mathcal{A})$ cannot be constant for n large enough.

Dirac [3] showed an elementary proof also exists: obviously, $r(n, \mathcal{A})$ is odd when $n = 2a$, $a \in \mathcal{A}$, and even, otherwise. Moreover, by using the technique of generating functions, he gave a short and elegant proof that

$$r^+(n, \mathcal{A}) = \#\{(a, a') \mid n = a + a', \quad a, a' \in \mathcal{A}, \quad a \leq a'\}$$

cannot be constant, either.

Ruzsa made an surprising example which shows that the above conjecture does not hold if one replaces $a + a'$ with $a + 2a'$.

Example of Ruzsa: Let

$$\mathcal{A} = \{a : a = \sum_{i=0}^{+\infty} \varepsilon_i 2^{2i}, \varepsilon_i = 0 \text{ or } 1\}.$$

Then, for $n \in \mathbb{N}$, the representation function

$$r_{1,2}(n, \mathcal{A}) = \#\{(a, a') \mid n = a + 2a', \quad a, a' \in \mathcal{A}\}$$

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is always 1.

Replacing base 2 by base k , Ruzsa's example still works, see earlier arguments of Moser [6], whose approach is also through generating functions.

More generally, Sárkozy and Sós [7] asked the following question on the representation function of multi-linear forms.

Question: For which (c_1, \dots, c_k) , can the representation function

$$r_{c_1, \dots, c_k}(n, \mathcal{A}) = \{(a_1, \dots, a_k) | c_1 a_1 + \dots + c_k a_k = n, a_1, \dots, a_k \in \mathcal{A}\}$$

be constant for n large enough.

Recently, Cilleruelo and Rué [2] gave a partial answer to the above question.

Theorem (Cilleruelo and Rué): Let $1 < c_1 < c_2$ and $\gcd(c_1, c_2) = 1$. There is no infinite subset \mathcal{A} of positive integers such that $r_{c_1, c_2}(n, \mathcal{A})$ is constant for n large enough.

Combining the earlier work of Moser [6], they completely solved Sárkozy and Sós's question for bilinear forms.

Every multilinear form $c_1 x_1 + \dots + c_k x_k$ can be uniquely written as

$$b_0(x_{0,1} + \dots + x_{0,e_0}) + b_1(x_{1,1} + \dots + x_{1,e_1}) + \dots + b_m(x_{m,1} + \dots + x_{m,e_m})$$

with

$$0 < b_0 < b_1 < \dots < b_m \text{ and } e_0, e_1, \dots, e_m > 0.$$

We call e_0, e_1, \dots, e_m the multiplicities of b_0, b_1, \dots, b_m , respectively. Denote

$$M = \{(b_0, e_0), \dots, (b_m, e_m)\}.$$

The representation function of n with respect to M , or equivalently with respect to the multi-linear form $c_1 x_1 + \dots + c_k x_k$, is the number of solutions of the equation

$$n = b_0(a_{0,1} + \dots + a_{0,e_0}) + \dots + b_m(a_{m,1} + \dots + a_{m,e_m})$$

with $a_{i,j} \in \mathcal{A}$. We denote this value by $r_M(n, \mathcal{A})$.

The main result of Rué [8] implied that $r_M(n, \mathcal{A})$ cannot be constant for n large enough if $\gcd(e_0, \dots, e_m) \geq 2$. The main tool which he used in the proof is from analytic combinatorics.

Remark 1.1. From Rué's [8] result, the case of $m = 0$ is clear. That is, for $m = 0$, $r_M(n, \mathcal{A})$ can be constant for n large enough if and only if $M = \{(1, 1)\}$. So from now on, we will always assume $m \geq 1$ unless specification.

In this paper, we will prove the following theorem .

Theorem 1.2. *Let $m \geq 1$. Assume there exists a prime p and a positive integer t such that $p^t \mid b_0$, but $p^t \nmid b_i$ for $1 \leq i \leq m$. Then, for any $\mathcal{A} \subseteq \mathbb{N}$, $r_M(n, \mathcal{A})$ cannot be constant for n large enough.*

Note that the conditions of Theorem 1.2 include the case: b_0, b_1, \dots, b_m are pairwise coprime. Therefore, the above theorem generalizes the theorem of Cilleruelo and Rué [2] from bilinear forms to multi-linear forms.

It should be noted that our method is different from theirs. For example, the approach of [2] makes use of complex analysis, but ours is purely

algebraic. The new ingredients in our proof are fractional power series and their logarithmic derivatives.

2. THE IDEA OF PROOF: TRANSLATION OF THE PROBLEM INTO GENERATING FUNCTIONS

For every set \mathcal{A} of non-negative integers, the generating function of \mathcal{A} is the formal power series $f_{\mathcal{A}}(x)$ defined as

$$f_{\mathcal{A}}(x) = \sum_{a \in \mathcal{A}} x^a.$$

In this way, the subsets of non-negative integers are in one to one correspondence to formal power series with coefficients 0 or 1.

The power series $f_{\mathcal{A}}(x)$ also defines an analytic function around $x = 0$. Indeed, if \mathcal{A} is finite, then $f_{\mathcal{A}}(x)$ is a polynomial. Otherwise, $f_{\mathcal{A}}(x)$ has radius of convergence $r = 1$ at $x = 0$.

We now translate the combinatorial problem in the language of generating functions. Let \mathcal{A} be a subset of non-negative integers and

$$M = \{(b_0, e_0), (b_1, e_1), \dots, (b_m, e_m)\}.$$

The following equation is fundamental:

$$\begin{aligned} & f_{\mathcal{A}}(x^{b_0})^{e_0} f_{\mathcal{A}}(x^{b_1})^{e_1} \dots f_{\mathcal{A}}(x^{b_m})^{e_m} \\ &= \sum_{a_{i,j} \in \mathcal{A}} x^{b_0(a_{0,1} + \dots + a_{0,e_0}) + \dots + b_m(a_{m,1} + \dots + a_{m,e_m})} \\ (2.1) \quad &= \sum_{n=0}^{+\infty} r_M(n, \mathcal{A}) x^n. \end{aligned}$$

Assume $r_M(n, \mathcal{A}) = c \neq 0$ for $n > n_0$. Then

$$\begin{aligned} \sum_{n=0}^{+\infty} r_M(n, \mathcal{A}) x^n &= \sum_{n=0}^{n_0} r_M(n, \mathcal{A}) x^n + \sum_{n=n_0+1}^{+\infty} c x^n \\ (2.2) \quad &= \sum_{n=0}^{n_0} r_M(n, \mathcal{A}) x^n + \frac{c x^{n_0+1}}{1-x} = \frac{P(x)}{1-x}, \end{aligned}$$

where $P(x)$ is a polynomial in $\mathbb{Z}[x]$ with $P(1) \neq 0$. Notice that $P(1) \neq 0$ is equivalent to $c \neq 0$.

Combining (2.1) and (2.2), we see that if $r_M(n, \mathcal{A}) = c \neq 0$ for n large enough, then $f(x) = f_{\mathcal{A}}(x)$ is a solution in $\mathbb{Z}[[x]]$ of the following equation

$$(2.3) \quad f(x^{b_0})^{e_0} f(x^{b_1})^{e_1} \dots f(x^{b_m})^{e_m} = \frac{P(x)}{1-x}$$

for some $P(x) \in \mathbb{Z}[x]$ with $P(1) \neq 0$.

Conversely, a solution $f(x) \in \mathbb{Z}[[x]]$ of (2.3) with coefficients in $\{0, 1\}$ defines, by the relation $f(x) = f_{\mathcal{A}}(x)$, a subset \mathcal{A} such that

$$r_M(n, \mathcal{A}) = c \neq 0$$

for n large enough.

Summing up, we get the following lemma.

Lemma 2.1. *There exists an infinite subset \mathcal{A} of non-negative integers such that $r_M(n, A)$ is a nonzero constant for n large enough if and only if there is a polynomial $P(x) \in \mathbb{Z}[x]$ with $P(1) \neq 0$ such that (2.3) has a solution $f(x) \in \mathbb{Z}[[x]]$ with coefficients $\in \{0, 1\}$.*

It is convenient to work with power series with constant term being 1. For subset \mathcal{A} of non-negative integers, this can be achieved by replacing \mathcal{A} by $\mathcal{A} - \min\{x | x \in \mathcal{A}\}$. Alternatively, for power series $f(x)$, this can be achieved by dividing $f_{\mathcal{A}}(x)$ by the lowest term. Obviously, this does not affect the solvability of equation (2.3). So, from now on, we always assume $0 \in \mathcal{A}$ and $f(0) = 1$ unless specification.

By lemma 2.1 and the above arguments, to prove Theorem 2.1, it is sufficient to prove the following theorem.

Theorem 2.2. *Let $m \geq 1$. Assume there exists a prime p and a positive integer t such that $p^t \mid b_0$, but $p^t \nmid b_i$ for $1 \leq i \leq m$. Then, for any $P(x) \in \mathbb{Z}[x]$ with $P(0) = 1$ and $P(1) \neq 0$, the equation*

$$(2.4) \quad f(x^{b_0})^{e_0} f(x^{b_1})^{e_1} \cdots f(x^{b_m})^{e_m} = \frac{P(x)}{1-x}$$

has no solution in $\mathbb{C}[[x]]$.

In the following, we will illustrate the idea of the proof, especially the motivation of fractional power series. Let us look at Moser's argument first.

Moser's argument: For each $k \geq 2$, Moser constructed an infinite set \mathcal{A} such that $r_{1,k}(n, \mathcal{A}) = 1$ for all $n \geq 0$ by solving the equation

$$f(x)f(x^k) = \sum_{n \geq 0}^{+\infty} x^n = \frac{1}{1-x}.$$

Writing it as

$$f(x) = \frac{1}{1-x} f(x^k)^{-1},$$

then iterate

$$\begin{aligned} f(x) &= \frac{1}{1-x} \left(\frac{1}{1-x^k} \right)^{-1} f(x^{k^2}) \\ &= \frac{1}{1-x} \left(\frac{1}{1-x^k} \right)^{-1} \left(\frac{1}{1-x^{k^2}} \right)^{-1} f(x^{k^3})^{-1} \\ &= \cdots \\ &= \prod_{i=0}^{j-1} \left(\frac{1}{1-x^{k^i}} \right)^{(-1)^i} \cdot f(x^{k^j})^{(-1)^j}. \end{aligned}$$

Letting $j \rightarrow +\infty$, we get

$$f(x) = \prod_{i=0}^{+\infty} \left(\frac{1}{1-x^{k^i}} \right)^{(-1)^i} = \prod_{i=0}^{+\infty} (1 + x^{k^{2i}} + x^{2k^{2i}} + \cdots + x^{(k-1)k^{2i}}).$$

By the uniqueness of k -adic representation of an integer, $f(x)$ is the generating function of the set

$$\mathcal{A} = \left\{ \sum_{i=0}^{+\infty} \varepsilon_i k^{2i}, \varepsilon_i \in \{0, 1, \dots, k-1\} \right\}.$$

Our initial approach is similar to Moser's argument. For simplicity, we will take the example, $M = \{(2, 1), (3, 1)\}$ to illustrate the ideas. From Lemma 2.1, we need to consider the equation

$$(2.5) \quad f(x^2)f(x^3) = \frac{P(x)}{1-x}.$$

with $P(0) = 1$ and $P(1) \neq 0$.

If Equation (2.5) has a solution $f = f_{\mathcal{A}}$ for some infinite subset \mathcal{A} of non-negative integers, then f defines an analytic function in the unit disk with $f(0) = 1$.

Let $0 < x < 1$ and $x^{\frac{1}{2}}$ be the positive square root of x . Substituting x by $x^{\frac{1}{2}}$ in (2.5), we get

$$f(x) = \frac{P(x^{\frac{1}{2}})}{1-x^{\frac{1}{2}}} f(x^{\frac{3}{2}})^{-1}.$$

Repeating Moser's arguments, we get

$$f(x) = \prod_{k=0}^{j-1} \left(\frac{P(x^{\frac{1}{2}(\frac{3}{2})^k})}{1-x^{\frac{1}{2}(\frac{3}{2})^k}} \right)^{(-1)^k} f(x^{(\frac{3}{2})^j})^{(-1)^j}.$$

Since $x^{\frac{1}{2}(\frac{3}{2})^j} \rightarrow 0$ and $f(x^{\frac{1}{2}(\frac{3}{2})^j}) \rightarrow 1$ as $j \rightarrow +\infty$, we obtain

$$(2.6) \quad f(x) = \prod_{k=0}^{+\infty} \left(\frac{P(x^{\frac{1}{2}(\frac{3}{2})^k})}{1-x^{\frac{1}{2}(\frac{3}{2})^k}} \right)^{(-1)^k},$$

for $0 < x < 1$.

Viewing $x^{\frac{1}{2}(\frac{3}{2})^k}$ as an analytic function defined in $\mathbb{C} - (-\infty, 0]$ with value 1 at $x = 1$, since for any positive integer n ,

$$\sum_{k=0}^{+\infty} |x^{n\frac{1}{2}(\frac{3}{2})^k}| \leq \sum_{k=0}^{+\infty} r^{n\frac{1}{2}(\frac{3}{2})^k} < +\infty, \text{ if } |x| \leq r < 1,$$

the infinite products

$$\prod_{k=0}^{+\infty} P(x^{\frac{1}{2}(\frac{3}{2})^{2k}}), \prod_{k=0}^{+\infty} P(x^{\frac{1}{2}(\frac{3}{2})^{2k+1}}), \prod_{k=0}^{+\infty} (1 - x^{\frac{1}{2}(\frac{3}{2})^{2k}}), \text{ and } \prod_{k=0}^{+\infty} (1 - x^{\frac{1}{2}(\frac{3}{2})^{2k+1}})$$

are absolutely and uniformly convergent in compact subsets of

$$D' = \{x \mid x \in \mathbb{C}, |x| < 1\} - \{x \mid x \in \mathbb{R}, -1 < x \leq 0\},$$

hence, analytic in D' (e.g., see Proposition 3.2 of Chapter 5 of [10]). Therefore, the right hand side of (2.6) is a meromorphic function in D' .

The analytic function f is determined by its values on the interval $(0, 1)$ (e.g., see Corollary 4.9 of Chapter 2 of [10]). Therefore, by (2.6),

$$(2.7) \quad f(x) = \prod_{k=0}^{+\infty} \left(\frac{P(x^{\frac{1}{2}(\frac{3}{2})^k})}{1 - x^{\frac{1}{2}(\frac{3}{2})^k}} \right)^{(-1)^k}$$

holds for all $x \in D'$.

However, from Equation (2.7), it seems that $f(x)$ can not be analytic around 0. This contradicts to the hypothesis that $f(x) = f_{\mathcal{A}}(x)$, which is analytic in the unit disk. The rigorous proof goes as follows.

A useful method to treat “infinite products” is taking its logarithmic derivative, which transforms “infinite products” to “infinite sums” (e.g., see Proposition 3.2 of Chapter 5 of [10]). So instead of considering $f(x)$, we look at $\frac{f'(x)}{f(x)}$. Since $f(0) = 1$, if $f(x)$ is analytic around zero, so is $\frac{f'(x)}{f(x)}$.

As $P(0) = 1$, we can assume

$$\frac{P(x)}{1-x} = \prod_i (1 - \alpha_i x)^{n_i}, \text{ with } \alpha_i \text{'s distinct.}$$

Denote

$$\frac{P(x)}{1-x} = G(x).$$

Then

$$(2.8) \quad x \frac{G'(x)}{G(x)} = \sum_i n_i \frac{-\alpha_i x}{1 - \alpha_i x} = - \sum_{n=1}^{+\infty} \sum_i n_i \alpha_i^n x^n.$$

From equation (2.7) and (2.8), we have

$$(2.9) \quad \begin{aligned} x \frac{f'(x)}{f(x)} &= \sum_{k=0}^{+\infty} (-1)^k \frac{G(x^{\frac{1}{2}(\frac{3}{2})^k})' x}{G(x^{\frac{1}{2}(\frac{3}{2})^k})} \\ &= \sum_{k=0}^{+\infty} (-1)^k \frac{1}{2} \left(\frac{3}{2}\right)^k \frac{G'(x^{\frac{1}{2}(\frac{3}{2})^k})}{G(x^{\frac{1}{2}(\frac{3}{2})^k})} x^{\frac{1}{2}(\frac{3}{2})^k} \\ &= - \sum_{k=0}^{+\infty} (-1)^k \frac{1}{2} \left(\frac{3}{2}\right)^k \sum_{n=1}^{+\infty} \sum_i n_i \alpha_i^n x^{n \frac{1}{2}(\frac{3}{2})^k} \end{aligned}$$

Note that since $|x| < 1$, $x^{\frac{1}{2}(\frac{3}{2})^k}$ goes to zero very fast as $k \rightarrow +\infty$. A routine argument, which we do not make here, shows that (2.9) are absolutely and uniformly convergent, in a small neighborhood of zero (inside D'). Therefore, we can take derivatives of (2.9) term by term (e.g., see Theorem 5.3 of Chapter 2 of [10]).

Taking derivatives of (2.9) of all order term by term and evaluating the derivatives at zero, one can see $\frac{x f'(x)}{f(x)}$ is analytic around zero if and only if the coefficient of x^λ in (2.9) is zero, whenever $\lambda \notin \mathbb{N}$.

Letting the coefficient of x^λ in (2.9) be 0, we get the equation

$$(2.10) \quad \sum_{\frac{1}{2}(\frac{3}{2})^k | \lambda} (-1)^k \frac{1}{2} \left(\frac{3}{2}\right)^k \sum_i n_i \alpha_i^{\lambda \cdot 2(\frac{2}{3})^k} = 0,$$

where

$$\frac{1}{2} \left(\frac{3}{2}\right)^k | \lambda \Leftrightarrow \lambda \cdot 2 \left(\frac{2}{3}\right)^k \in \mathbb{N}.$$

Finally, we succeed to prove for all $\lambda \notin \mathbb{N}$, Equations (2.10) have no common solution α'_i 's.

After that, we realized that $f(x)$ and $x \frac{f'(x)}{f(x)}$ (see Equations (2.7), (2.9)) can be viewed as some generalized formal series, which we call fractional power series. Then everything can be computed formally in the ring of fractional power series. In the rest of paper, we will use fractional power series other than analytic functions since the convergence of the former ones are much simpler than the latter ones.

The above arguments explain the motivation of using fractional power series. As far as we know, the notion of fractional power series do not appear in the literature. So they will be defined and discussed in detail in Section 3. Generally speaking, fractional power series behave like formal power series.

After the preparation of section 3, we begin to prove the main result of this paper, Theorem 2.2. The proof is actually direct, but it is rather long. So We had better divide it into several steps. The plan of the proof will be described in detail at the beginning of section 4, after we introduce some basic notations. Section 4 and section 5 provide all the ingredients of the proof. Finally, we prove Theorem 2.2 in section 6.

At last, we discuss the question of Sárközy and Sós in section 7. We will give a conjectural answer in the case that all the coefficients of linear forms are positive.

3. FRACTIONAL POWER SERIES

In this section, we introduce the concept of fractional power series and basic operations of them, including their convergence, derivatives, infinite products and logarithmic derivatives, etc. We also prove their basic properties. These are fundamental to our later computations.

Let $\theta_1, \dots, \theta_m > 1$ be distinct real numbers and $b \in \mathbb{N}$ be a positive integer. Define $\mathbb{Z}_{\geq 0}[x_1, \dots, x_m]$ to be the set of polynomials of x_1, \dots, x_m with coefficients of non-negative integers. Define

$$\Lambda = \left\{ \frac{1}{b} F(\theta_1, \dots, \theta_m) \mid F \in \mathbb{Z}_{\geq 0}[x_1, \dots, x_m] \right\}.$$

We call Λ the lattice associated to $(b; \theta_1, \dots, \theta_m)$.

Proposition 3.1. *Let Λ be defined as above. Then*

- (1) Λ is discrete, i.e., $\forall M > 0$, $\{\lambda \in \Lambda \mid \lambda < M\}$ is a finite set.
- (2) If $\lambda, \lambda' \in \Lambda$, then $\lambda + \lambda' \in \Lambda$.
- (3) If $\lambda \in \Lambda$, then $\theta_i \lambda \in \Lambda$ for $i = 1, \dots, m$.
- (4) $\mathbb{Z}_{\geq 0} \subseteq \Lambda$, where $\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers.

Proof. We only prove (1). Let $F \in \mathbb{Z}_{\geq 0}[x_1, \dots, x_m]$. Denote the total degree of F with respect to x_1, \dots, x_m by d . Let C be an arbitrary coefficient of F . If

$$(3.1) \quad \frac{1}{b} F(\theta_1, \dots, \theta_m) \leq M,$$

then

$$\begin{aligned} (\min\{\theta_1, \dots, \theta_m\})^d &\leq F(\theta_1, \dots, \theta_m) \leq bM, \\ 0 &\leq C \leq F(\theta_1, \dots, \theta_m) \leq bM. \end{aligned}$$

So there are only finitely many F satisfying Equation (3.1). \square

Definition 3.2. The formal series

$$\sum_{\lambda \in \Lambda} c_\lambda x^\lambda, \text{ with } c_\lambda \in \mathbb{C},$$

are called fractional power series with respect to Λ , or $(b; \theta_1, \dots, \theta_m)$. Define

$$\sum_{\lambda \in \Lambda} c_\lambda x^\lambda = \sum_{\lambda \in \Lambda} c'_\lambda x^\lambda$$

if and only if $c_\lambda = c'_\lambda$ for all $\lambda \in \Lambda$. Denote $\mathbb{C}[[x^\Lambda]]$ to be the set of all fractional power series with respect to Λ .

The following definition makes $\mathbb{C}[[x^\Lambda]]$ a commutative ring with unit element.

Definition 3.3. For

$$\sum_{\lambda \in \Lambda} c_\lambda x^\lambda, \sum_{\lambda \in \Lambda} c'_\lambda x^\lambda \in \mathbb{C}[[x^\Lambda]],$$

their sum and product are defined as

$$\begin{aligned} \sum_{\lambda \in \Lambda} c_\lambda x^\lambda + \sum_{\lambda \in \Lambda} c'_\lambda x^\lambda &= \sum_{\lambda \in \Lambda} (c_\lambda + c'_\lambda) x^\lambda, \\ \sum_{\lambda \in \Lambda} c_\lambda x^\lambda \cdot \sum_{\lambda \in \Lambda} c'_\lambda x^\lambda &= \sum_{\lambda \in \Lambda} \left(\sum_{\mu + \nu = \lambda} c_\mu c'_\nu \right) x^\lambda. \end{aligned}$$

We call $\mathbb{C}[[x^\Lambda]]$ the ring of fractional power series with respect to

$$\Lambda \text{ or } (b; \theta_1, \dots, \theta_m).$$

By (3) of Proposition 3.1, we know that

$$\text{for any } \lambda \in \Lambda, \text{ the sum } \sum_{\mu + \nu = \lambda} c_\mu c_\nu$$

is a finite sum, so the multiplication of two elements of $\mathbb{C}[[x^\Lambda]]$ is well-defined. It is easily seen that x^0 is the unit element of $\mathbb{C}[[x^\Lambda]]$. Moreover, by (4) of Proposition 3.1, we have

$$\mathbb{C}[[x]] \subseteq \mathbb{C}[[x^\Lambda]].$$

Remark 3.4. If $\{\theta_1, \dots, \theta_m\} = \emptyset$, then

$$\mathbb{C}[[x^\Lambda]] = \left\{ \sum_{n \geq 0} c_n x^{\frac{n}{b}} \mid c_n \in \mathbb{C} \right\} = \mathbb{C}[[x^{\frac{1}{b}}]].$$

If $f \in \mathbb{C}[[x^{\frac{1}{b}}]]$ for some $b \geq 1$, then f is called a fractional power series by Stanley (see page 161 of [9]). So Definition 3.2 can be viewed as a generalization of Stanley's definition.

Generally speaking, $\mathbb{C}[[x^\Lambda]]$ has many properties similar to $\mathbb{C}[[x]]$. For example, we can define metrics both on $\mathbb{C}[[x^\Lambda]]$ and $\mathbb{C}[[x]]$, which make them complete metric spaces.

Definition 3.5. Let $f = \sum_{\lambda \in \Lambda} c_\lambda x^\lambda \in \mathbb{C}[[x^\Lambda]]$. The order of f , denoted by $\text{ord} f$, is defined as follows:

$$\text{ord} f = \begin{cases} \min_{c_\lambda \neq 0} \{\lambda\}, & \text{if } f \neq 0; \\ +\infty, & \text{if } f = 0. \end{cases}$$

We have the following proposition.

Proposition 3.6. Let $f, g \in \mathbb{C}[[x^\Lambda]]$. Then

- (1) $\text{ord}(f + g) \geq \min\{\text{ord} f, \text{ord} g\}$.
- (2) $\text{ord}(f \cdot g) = \text{ord} f + \text{ord} g$.
- (3) $\text{ord} f = +\infty \Leftrightarrow f = 0$.

Fix some real number $\beta \in (0, 1)$.

Definition 3.7. Let $f \in \mathbb{C}[[x^\Lambda]]$. The valuation of f , denoted by $|f|$, is equal to $\beta^{\text{ord} f}$.

Corresponding to Proposition 3.6, we have

Proposition 3.8. Let $f, g \in \mathbb{C}[[x^\Lambda]]$. Then

- (1) $|f + g| \leq \max\{|f|, |g|\}$.
- (2) $|f \cdot g| = |f||g|$.
- (3) $|f| = 0 \Leftrightarrow f = 0$.

Given two elements $f, g \in \mathbb{C}[[x^\Lambda]]$, their distance is defined as

$$d(f, g) = |f - g|.$$

By Proposition 3.8, $\mathbb{C}[[x^\Lambda]]$ is really a metric space. And (1) of Proposition 3.8 is usually called the strong triangle inequality.

A sequence $\{f_n\}_{n \in \mathbb{N}}$ is convergent to f if and only if

$$\lim_{n \rightarrow +\infty} d(f_n, f) = 0,$$

or equivalently,

$$\lim_{n \rightarrow +\infty} \text{ord}(f_n - f) = +\infty.$$

In this case, we denote

$$f = \lim_{n \rightarrow +\infty} f_n.$$

The following proposition shows that $\mathbb{C}[[x^\Lambda]]$ is a complete metric space, i.e., every cauchy sequence converges.

Proposition 3.9. *If $\lim_{n \rightarrow +\infty} \text{ord}(f_{n+1} - f_n) = +\infty$, then there exists $f \in \mathbb{C}[[x^\Lambda]]$ such that $f = \lim_{n \rightarrow +\infty} f_n$.*

Proof. Let

$$f_n = \sum_{\lambda \in \Lambda} c_{n,\lambda} x^\lambda.$$

Since

$$\lim_{n \rightarrow +\infty} \text{ord}(f_{n+1} - f_n) = +\infty,$$

for any $\lambda \in \Lambda$, there exists $N \in \mathbb{N}$ such that

$$\text{ord}(f_{n+1} - f_n) > \lambda \text{ for } n > N.$$

This implies that

$$c_{n+1,\lambda} = c_{n,\lambda}, \text{ when } n > N,$$

that is, for λ being fixed, the sequence $c_{n,\lambda}$ is constant for n large enough. Therefore, let

$$f = \sum_{\lambda \in \Lambda} \left(\lim_{n \rightarrow +\infty} c_{n,\lambda} \right) x^\lambda.$$

Then $\text{ord}(f - f_n) > \lambda$ if $n > N$. Therefore

$$\lim_{n \rightarrow +\infty} f_n = f.$$

□

For $f = \sum_{\lambda \in \Lambda} c_\lambda x^\lambda \in \mathbb{C}[[x^\Lambda]]$, denote c_0 by $f(0)$.

Corollary 3.10. *f is invertible if and only if $f(0) \neq 0$.*

Proof. If there exists $g \in \mathbb{C}[[x^\Lambda]]$ such that $f \cdot g = 1$, then $f(0)g(0) = 1$. Therefore, $f(0) \neq 0$.

Conversely, assume $f(0) \neq 0$. Write

$$f = f(0)(1 + h) \text{ with } \text{ord}(h) > 0.$$

By Proposition 3.9,

$$1 + \sum_{i=1}^{+\infty} (-1)^i h^i$$

converges. Therefore,

$$f^{-1} = f(0)^{-1} \left(1 + \sum_{i=1}^{+\infty} (-1)^i h^i \right) \in \mathbb{C}[[x^\Lambda]].$$

□

Corollary 3.11. *For $n \geq 1$, assume $\text{ord} f_n > 0$ and $\lim_{n \rightarrow +\infty} \text{ord} f_n = +\infty$.*

Then the infinite product $\prod_{n=1}^{+\infty} (1 + f_n)$ converges.

Proof. Since

$$\prod_{n=1}^{m+1} (1 + f_n) - \prod_{n=1}^m (1 + f_n) = f_{m+1} \prod_{n=1}^m (1 + f_n),$$

its order equals to $\text{ord} f_{m+1}$. By the assumption,

$$\lim_{m \rightarrow +\infty} \text{ord} \left(\prod_{n=1}^{m+1} (1 + f_n) - \prod_{n=1}^m (1 + f_n) \right) = +\infty.$$

By Proposition 3.9, we get the desired result. \square

Let

$$f(x) = \sum_{\lambda \in \Lambda} c_\lambda x^\lambda \in \mathbb{C}[[x^\Lambda]].$$

Then

$$\sum_{\lambda \in \Lambda} c_\lambda x^{\lambda \theta_i} \in \mathbb{C}[[x^\Lambda]],$$

by (3) of Proposition 3.1, where $i = 1, \dots, m$. Then define

$$f(x^{\theta_i}) = \sum_{\lambda \in \Lambda} c_\lambda x^{\lambda \theta_i}.$$

This can be viewed as changing variable “ x ” by “ x^θ ”. It is easy to see that the map $f(x) \mapsto f(x^{\theta_i})$ is a continuous ring homomorphism of $\mathbb{C}[[x^\Lambda]]$.

Definition 3.12. Let

$$f(x) = \sum_{\lambda \in \Lambda} c_\lambda x^\lambda \in \mathbb{C}[[x^\Lambda]].$$

Define the derivative of f by

$$xf'(x) = \sum_{\lambda \in \Lambda} \lambda c_\lambda x^\lambda \in \mathbb{C}[[x^\Lambda]].$$

Remark 3.13. In Definition 3.12, we multiply the usual derivative f' by x to make sure $xf' \in \mathbb{C}[[x^\Lambda]]$.

Proposition 3.14. *Let $f, g \in \mathbb{C}[[x^\Lambda]]$. Then*

- (1) $x(f + g)' = xf' + xg'$.
- (2) $x(f \cdot g)' = xf' \cdot g + f \cdot xg'$.
- (3) $x(f(x^{\theta_i}))' = \theta_i(xf')(x^{\theta_i})$, for $i = 1, \dots, m$.
- (4) $\lim_{n \rightarrow +\infty} (xf'_n) = x(\lim_{n \rightarrow +\infty} f_n)'$, if $\lim_{n \rightarrow +\infty} f_n$ exists.

Definition 3.15. Let

$$f(x) = \sum_{\lambda \in \Lambda} c_\lambda x^\lambda \in \mathbb{C}[[x^\Lambda]] \text{ with } f(0) \neq 0.$$

We call $\frac{xf'}{f}$ the logarithmic derivative of f .

Remark 3.16. If $f(0) = 0$, from Corollary 3.10, f is not invertible. Hence, $\frac{xf'}{f}$ may not belong to $\mathbb{C}[[x^\Lambda]]$. This is different from the case of $\mathbb{C}[[x]]$.

By Proposition 3.14, we have

Proposition 3.17. Let $f, g \in \mathbb{C}[[x^\Lambda]]$. Then

$$\begin{aligned} (1) \quad & \frac{x(fg)'}{fg} = \frac{xf'}{f} + \frac{xg'}{g} \\ (2) \quad & \frac{x(f(x^{\theta_i}))'}{f(x^{\theta_i})} = \theta_i \left(\frac{xf'}{f} \right) (x^{\theta_i}) \text{ for } i = 1, \dots, m. \end{aligned}$$

The following proposition shows that the logarithmic derivative transforms infinite products to infinite sums.

Proposition 3.18. For $n \geq 1$, assume $\text{ord} f_n > 0$ and $\lim_{n \rightarrow +\infty} \text{ord} f_n = +\infty$.

Then

$$x \left(\prod_{n=1}^{+\infty} (1 + f_n) \right)' \left(\prod_{n=1}^{+\infty} (1 + f_n) \right)^{-1} = \sum_{n=1}^{+\infty} x f_n' (1 + f_n)^{-1}.$$

Proof. By (1) of Proposition 3.17, we have

$$(3.2) \quad x \left(\prod_{n=1}^N (1 + f_n) \right)' \cdot \left(\prod_{n=1}^N (1 + f_n) \right)^{-1} = \sum_{n=1}^N x f_n' (1 + f_n)^{-1}.$$

By (4) of Proposition 3.14, we get

$$\lim_{N \rightarrow +\infty} x \left(\prod_{n=1}^N (1 + f_n) \right)' = x \left(\prod_{n=1}^{+\infty} (1 + f_n) \right)'.$$

Since

$$\lim_{N \rightarrow +\infty} \prod_{n=1}^N (1 + f_n)^{-1} = \left(\prod_{n=1}^{+\infty} (1 + f_n) \right)^{-1},$$

letting $N \rightarrow +\infty$ in (3.2), we get the desired result. \square

Proposition 3.19. Let $f \in \mathbb{C}[[x^\Lambda]]$ with $f(0) \neq 0$. Then

$$f \in \mathbb{C}[[x]] \Leftrightarrow \frac{xf'}{f} \in \mathbb{C}[[x]].$$

Proof. We only prove the “if” part. Assume

$$\frac{xf'}{f} = \sum_{n=1}^{+\infty} c_n x^n \in \mathbb{C}[[x]].$$

Let

$$g = \exp\left(\sum_{n=1}^{\infty} \frac{c_n}{n} x^n\right) = \sum_{m=0}^{+\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} \frac{c_n}{n} x^n\right)^m.$$

By (4) of Proposition 3.14, we have

$$xg' = \exp\left(\sum_{n=1}^{+\infty} \frac{c_n}{n} x^n\right) \sum_{n=1}^{+\infty} c_n x^n.$$

Therefore,

$$\frac{xg'}{g} = \sum_{n=1}^{+\infty} c_n x^n = \frac{xf'}{f}.$$

Since

$$x\left(\frac{g}{f}\right)' \left(\frac{g}{f}\right)^{-1} = \frac{xf'}{f} - \frac{xg'}{g} = 0,$$

we get $\frac{g}{f}$ is a constant. Thus $f \in \mathbb{C}[[x]]$. □

The following power series are well-known.

$$\begin{aligned} \exp(x) &= \sum_{n=0}^{+\infty} \frac{x^n}{n!}, \\ \log(1+x) &= \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} x^n}{n}, \\ (1+x)^\alpha &= \sum_{n=0}^{+\infty} \binom{\alpha}{n} x^n, \text{ where } \alpha \in \mathbb{C}, \\ \text{and } \binom{\alpha}{n} &= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n(n-1)\cdots 1}. \end{aligned}$$

Let $f \in \mathbb{C}[[x^\Lambda]]$ with $\text{ord}(f) > 0$. Then we can define $\exp(f)$, $\log(1+f)$, $(1+f)^\alpha$ by replacing x with f in the above expressions. It is easy to see the equalities which hold for $\exp(x)$, $\log(x)$, $(1+x)^\alpha$ also hold for $\exp(f)$, $\log(1+f)$, $(1+f)^\alpha$. For example, we have

$$\begin{aligned} ((1+f)^\alpha)^\beta &= (1+f)^{\alpha\beta}, \\ (1+f)^\alpha (1+f)^\beta &= (1+f)^{\alpha+\beta}, \\ (1+f)^\alpha (1+g)^\alpha &= (1+f+g+fg)^\alpha, \\ x((1+f)^\alpha)' &= \alpha(1+f)^{\alpha-1} x f' \end{aligned}$$

where $\text{ord}(f), \text{ord}(g) > 0$, and $\alpha, \beta \in \mathbb{C}$.

Finally, we prove the following proposition.

Proposition 3.20. *For $n \geq 1$, let $\alpha_n, \beta_n \in \mathbb{C}$. Then*

$$(3.3) \quad \prod_{n=1}^{+\infty} (1-x^n)^{\alpha_n} = \prod_{n=1}^{+\infty} (1-x^n)^{\beta_n}$$

if and only if $\alpha_n = \beta_n$ for all $n \geq 1$.

Proof. Taking the logarithmic derivatives of (3.3), we get

$$(3.4) \quad -\sum_{n=1}^{+\infty} \frac{\alpha_n n x^n}{1-x^n} = -\sum_{n=1}^{+\infty} \frac{\beta_n n x^n}{1-x^n}.$$

Comparing the coefficients of the lowest terms of (3.4), we get $\alpha_1 = \beta_1$. Then

$$(3.5) \quad -\sum_{n=2}^{+\infty} \frac{\alpha_n n x^n}{1-x^n} = -\sum_{n=2}^{+\infty} \frac{\beta_n n x^n}{1-x^n}.$$

Repeating the same procedure, we obtain $\alpha_2 = \beta_2, \dots, \alpha_n = \beta_n, \dots$. This concludes the proof. \square

4. SOLVING EQUATION WITH FRACTIONAL POWER SERIES

From now on, the following notations will be used unless specification.

- $2 \leq b = b_0 < b_1 < \dots < b_m$, positive integers.
- $\theta_1 = \frac{b_1}{b_0}, \dots, \theta_m = \frac{b_m}{b_0}$.
- Λ , the lattice associated to $(b; \theta_1, \dots, \theta_m)$.
- $\mathbb{C}[[x^\Lambda]]$, the ring of fractional power series with respect to Λ .
- $e = e_0, e_1, \dots, e_m$, positive integers.
- $\nu_1 = \frac{e_1}{e_0}, \dots, \nu_m = \frac{e_m}{e_0}$.
- $\mathbb{N}' = \{N \in \mathbb{N} \mid p|N \Rightarrow p|b_0 b_1 \dots b_m\}$
- $\mathbb{Q}' = \{\frac{n}{m} \mid n, m \in \mathbb{N}'\}$.
- $\mathbb{Q}_b = \{\frac{n}{b^t} \mid n \in \mathbb{N}, t \in \mathbb{Z}, t \geq 0\}$.
- $\mathbb{Q}'_b = \{\frac{n}{b^t} \mid n \in \mathbb{N}', t \in \mathbb{Z}, t \geq 0\}$.
- $G(x) = \prod_{i=1}^t (1 - \alpha_i x)^{n_i}$, with α_i 's distinct, nonzero.
- $S = \{\alpha_i \mid \alpha_i^n = 1 \text{ for some } n \in \mathbb{N}', G(\frac{1}{\alpha_i}) = 0\}$.
- $H(x) = \prod_{\alpha_i \in S} (1 - \alpha_i x)^{n_i}$ the \mathbb{N}' -cyclotomic part of $G(x)$.
- $P(x) \in \mathbb{Z}[x]$, $P(0) = 1$, $P(1) \neq 0$.
- $\lambda \mid \mu \Leftrightarrow \mu \lambda^{-1} \in \mathbb{N}$, where $\lambda, \mu \in \mathbb{Q}$, and $\lambda, \mu > 0$.

In this section, we will prove that the equation

$$(4.1) \quad f(x^{b_0})^{e_0} f(x^{b_1})^{e_1} \dots f(x^{b_m})^{e_m} = G(x)$$

has a unique solution $f(x)$ in $\mathbb{C}[[x^\Lambda]]$ with $f(0) = 1$.

By taking the logarithmic derivative of f , we give a criterion for f being a power series. As a corollary, we show if (4.1) has a power series solution, then the equation

$$(4.2) \quad g(x^{b_0})^{e_0} g(x^{b_1})^{e_1} \cdots g(x^{b_m})^{e_m} = H(x)$$

has a power series solution $g(x)$ with $g(0) = 1$, where $H(x)$ is the \mathbb{N}' -cyclotomic part of $G(x)$.

Moreover, if $H(x)$ has the following form

$$(4.3) \quad H(x) = \prod_{d \in \mathbb{N}'} (1 - x^d)^{m_d}, \quad m_d \in \mathbb{Z}, \quad m_d = 0 \text{ for } d \gg 0,$$

then the power series solution of (4.2) (if it exists) can be explicitly given by

$$(4.4) \quad g(x) = \prod_{d \in \mathbb{N}'} (1 - x^d)^{g_d}, \quad g_d \in \mathbb{Q}.$$

Finally, under some conditions on b_0, b_1, \dots, b_m , we show $g(x)$ is almost rational, that is,

$$g_d = 0 \text{ for } d \gg 0,$$

in Equation (4.4) (see Theorem 4.7).

In the next section, under certain conditions on $H(x)$, we will show that $g(x)$, the solution of (4.2), can not be almost rational (see Theorem 5.1). A contradiction!

This finally leads to the proof of Theorem 2.2 if we apply the above results to the case

$$(4.5) \quad G(x) = \frac{P(x)}{1 - x}.$$

and assume the existence of a prime p and a positive integer t such that $p^t \mid b_0$, but $p^t \nmid b_i$ for $1 \leq i \leq m$.

Theorem 4.1. *The equation*

$$(4.6) \quad f(x^{b_0})^{e_0} f(x^{b_1})^{e_1} \cdots f(x^{b_m})^{e_m} = G(x)$$

has a unique solution $f(x) \in \mathbb{C}[[x^\Lambda]]$ with $f(0) = 1$. In fact,

$$(4.7) \quad f(x) = \prod_{k=0}^{+\infty} \left(\prod_{1 \leq i_1, \dots, i_k \leq m} G(x^{b^{-1}\theta_{i_1}, \dots, \theta_{i_k}})^{e^{-1}\nu_{i_1}, \dots, \nu_{i_k}} \right)^{(-1)^k}.$$

Proof. Existence: Recall $b = b_0, e = e_0$. Substituting x by $x^{\frac{1}{b}}$ and taking the e -th root of both sides of Equation (4.6), we get

$$(4.8) \quad f(x) f(x^{\theta_1})^{\nu_1} \cdots f(x^{\theta_m})^{\nu_m} = G(x^{\frac{1}{b}})^{\frac{1}{e}}.$$

Then

$$(4.9) \quad f(x) = G(x^{\frac{1}{b}})^{\frac{1}{e}} (f(x^{\theta_1})^{\nu_1} \cdots f(x^{\theta_m})^{\nu_m})^{-1}.$$

Let $\mathcal{F}(f) = f(x^{\theta_1})^{\nu_1} \cdots f(x^{\theta_m})^{\nu_m}$ and view it as an operator on $\mathbb{C}[[x^\Lambda]]$. Obviously, \mathcal{F} is multiplicative. Rewrite Equation (4.9) as

$$(4.10) \quad f(x) = G(x^{\frac{1}{b}})^{\frac{1}{e}} \mathcal{F}(f)^{-1}.$$

Then iterate

$$\begin{aligned}
 f(x) &= G(x^{\frac{1}{b}})^{\frac{1}{e}} \mathcal{F}(G(x^{\frac{1}{b}})^{\frac{1}{e}})^{-1} \mathcal{F}^2(f) \\
 &= \dots\dots \\
 (4.11) \quad &= \prod_{k=0}^{n-1} \mathcal{F}^k(G(x^{\frac{1}{b}})^{\frac{1}{e}})^{(-1)^k} \mathcal{F}^n(f)^{(-1)^n}.
 \end{aligned}$$

Since

$$\mathcal{F}^n(f) = \prod_{1 \leq i_1, \dots, i_n \leq m} f(x^{\theta_{i_1} \dots \theta_{i_n}})^{\nu_{i_1} \dots \nu_{i_n}},$$

we have $\lim_{n \rightarrow +\infty} \mathcal{F}^n(f) = 1$.

Letting $n \rightarrow +\infty$ in Equation (4.11), by Corollary 3.11, we get

$$\begin{aligned}
 f(x) &= \prod_{k=0}^{+\infty} \mathcal{F}^k(G(x^{\frac{1}{b}})^{\frac{1}{e}})^{(-1)^k} \\
 (4.12) \quad &= \prod_{k=0}^{+\infty} \left(\prod_{1 \leq i_1, \dots, i_k \leq m} G(x^{b^{-1}\theta_{i_1} \dots \theta_{i_k}})^{e^{-1}\nu_{i_1} \dots \nu_{i_k}} \right)^{(-1)^k}.
 \end{aligned}$$

Substituting (4.12) into (4.8), we get

$$f \cdot \mathcal{F}(f) = \prod_{k=0}^{+\infty} \mathcal{F}^k(G(x^{\frac{1}{b}})^{\frac{1}{e}})^{(-1)^k} \prod_{k=0}^{+\infty} \mathcal{F}^{k+1}(G(x^{\frac{1}{b}})^{\frac{1}{e}})^{(-1)^k} = G(x^{\frac{1}{b}})^{\frac{1}{e}}.$$

So, (4.12) is really a solution of (4.6).

Uniqueness: Assume $f^* \in \mathbb{C}[[x^\Lambda]]$ is another solution of (4.6) with $f^*(0) = 1$, then

$$(4.13) \quad \frac{f}{f^*}(x^{b_0})^{e_0} \frac{f}{f^*}(x^{b_1})^{e_1} \dots \frac{f}{f^*}(x^{b_m})^{e_m} = 1.$$

Write

$$\frac{f}{f^*} = 1 + c_\mu x^\mu + \sum_{\lambda > \mu} c_\lambda x^\lambda$$

with $c_\mu \neq 0$. Then the right hand side of Equation (4.13) is

$$1 + e_0 c_\mu x^{b_0 \mu} + \text{“higher order terms”}.$$

A contradiction. So $f^* = f$. □

Theorem 4.2. *The solution (4.7) is a power series if and only if for any $\lambda \in \mathbb{Q}'_b$ satisfying $\lambda \notin \mathbb{N}'$, any $u \in \mathbb{N}$, $(u, b_0 \dots b_m) = 1$, and any $\beta \in \mathbb{C}^*$, the following equation*

$$\begin{aligned}
 (4.14) \quad &\sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1} \dots \theta_{i_k} | \lambda} \frac{\nu_{i_1} \dots \nu_{i_k}}{e} \frac{\theta_{i_1} \dots \theta_{i_k}}{b} \\
 &\quad \times \sum_{\substack{\lambda b \theta_{i_1}^{-1} \dots \theta_{i_k}^{-1} b_0 \dots b_m \\ \alpha_i = \beta}} n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1} \dots \theta_{i_k}^{-1}} = 0
 \end{aligned}$$

holds, where $b^{-1}\theta_{i_1} \cdots \theta_{i_k} \mid \lambda$ means their quotient is a positive integer.

Proof. We will compute the logarithmic derivative of f by equation (4.7).

Since

$$G(x) = \prod_i (1 - \alpha_i x)^{n_i},$$

we have

$$(4.15) \quad \frac{xG(x)'}{G(x)} = \sum_i \frac{-n_i \alpha_i x}{1 - \alpha_i x} = - \sum_{n=1}^{+\infty} \left(\sum_i n_i \alpha_i^n \right) x^n.$$

By (2) of Proposition 3.17, for $1 \leq i_1, \dots, i_k \leq m$,

$$(4.16) \quad \frac{xG(x^{b^{-1}\theta_{i_1} \cdots \theta_{i_k}})'}{G(x^{b^{-1}\theta_{i_1} \cdots \theta_{i_k}})} = - \frac{\theta_{i_1} \cdots \theta_{i_k}}{b} \sum_{n=1}^{+\infty} \left(\sum_i n_i \alpha_i^n \right) x^{nb^{-1}\theta_{i_1} \cdots \theta_{i_k}}.$$

By Proposition 3.18, Equation (4.7) and Equation (4.16),

$$(4.17) \quad \begin{aligned} \frac{xf'}{f} &= \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \frac{\nu_{i_1} \cdots \nu_{i_k}}{e} \frac{xG(x^{b^{-1}\theta_{i_1} \cdots \theta_{i_k}})'}{G(x^{b^{-1}\theta_{i_1} \cdots \theta_{i_k}})} \\ &= - \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \frac{\nu_{i_1} \cdots \nu_{i_k}}{e} \frac{\theta_{i_1} \cdots \theta_{i_k}}{b} \\ &\quad \times \sum_{n=1}^{+\infty} \left(\sum_i n_i \alpha_i^n \right) x^{nb^{-1}\theta_{i_1} \cdots \theta_{i_k}}. \end{aligned}$$

Let $\mu \in \mathbb{Q}$. The coefficient of x^μ in $\frac{xf'}{f}$ is

$$(4.18) \quad - \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1} \cdots \theta_{i_k} \mid \mu} \frac{\nu_{i_1} \cdots \nu_{i_k}}{e} \frac{\theta_{i_1} \cdots \theta_{i_k}}{b} \sum_i n_i \alpha_i^{\mu b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}}.$$

By Proposition 3.19, $f \in \mathbb{C}[[x]]$ if and only if $\frac{xf'}{f} \in \mathbb{C}[[x]]$. Therefore, $f \in \mathbb{C}[[x]]$ if and only if Equation (4.18) is zero for all $\mu \in \mathbb{Q} - \mathbb{N}$.

If $\mu \notin \mathbb{Q}_b$, Equation (4.18) is automatically zero.

For $\mu \in \mathbb{Q}_b - \mathbb{N}$, it can be uniquely written as

$$(4.19) \quad \mu = \lambda \cdot u, \text{ where } \lambda \in \mathbb{Q}'_b - \mathbb{N}', \text{ and } u \in \mathbb{N} \text{ s.t. } (u, b_0 \cdots b_m) = 1.$$

Substituting (4.19) into (4.18), we get the coefficient of x^μ is

$$(4.20) \quad \begin{aligned} &\sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1} \cdots \theta_{i_k} \mid \lambda u} \frac{\nu_{i_1} \cdots \nu_{i_k}}{e} \frac{\theta_{i_1} \cdots \theta_{i_k}}{b} \sum_i n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}} \\ &= \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1} \cdots \theta_{i_k} \mid \lambda} \frac{\nu_{i_1} \cdots \nu_{i_k}}{e} \frac{\theta_{i_1} \cdots \theta_{i_k}}{b} \sum_i n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}}, \end{aligned}$$

since $\lambda \in \mathbb{Q}'_b$ and $u \in \mathbb{N}$, s.t. $(u, b_0 \cdots b_m) = 1$.

Therefore, $f \in \mathbb{C}[[x]]$ if and only if

$$(4.21) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1} \cdots \theta_{i_k} | \lambda} \frac{\nu_{i_1} \cdots \nu_{i_k}}{e} \frac{\theta_{i_1} \cdots \theta_{i_k}}{b} \sum_i n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}} = 0$$

for all $\lambda \in \mathbb{Q}'_b - \mathbb{N}'$ and all $u \in \mathbb{N}$, s.t. $(u, b_0 \cdots b_m) = 1$.

In Equation (4.21), substituting u with $u + nb_0 \cdots b_m$, we get

$$(4.22) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1} \cdots \theta_{i_k} | \lambda} \frac{\nu_{i_1} \cdots \nu_{i_k}}{e} \frac{\theta_{i_1} \cdots \theta_{i_k}}{b} \times \sum_i n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}} (\alpha_i^{\lambda b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} b_0 \cdots b_m})^n = 0.$$

In Equation (4.22), fixed u , λ and letting n vary, we obtain a family of infinite equations indexed by $n \in \mathbb{N}$:

$$(4.23) \quad \sum_{\beta} \beta^n \left(\sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1} \cdots \theta_{i_k} | \lambda} \frac{\nu_{i_1} \cdots \nu_{i_k}}{e} \frac{\theta_{i_1} \cdots \theta_{i_k}}{b} \times \sum_{\substack{\lambda b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} b_0 \cdots b_m \\ \alpha_i = \beta}} n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}} \right) = 0.$$

In (4.23), since β s are distinct, the Vandermonde determinant

$$\det(\beta^n)_{\beta, n} \neq 0.$$

Therefore, the coefficient of β^n in (4.23) should be zero, that is,

$$(4.24) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1} \cdots \theta_{i_k} | \lambda} \frac{\nu_{i_1} \cdots \nu_{i_k}}{e} \frac{\theta_{i_1} \cdots \theta_{i_k}}{b} \times \sum_{\substack{\lambda b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} b_0 \cdots b_m \\ \alpha_i = \beta}} n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}} = 0,$$

for all $\lambda \in \mathbb{Q}'_b$, $\lambda \notin \mathbb{N}'$; $u \in \mathbb{N}$, $(u, b_0 \cdots b_m) = 1$; and $\beta \in \mathbb{C}^*$.

It is easy to see that Equation (4.24) implies Equation (4.21). Thus, the proof is complete. \square

Corollary 4.3. *If Equation (4.1) has a solution $f(x) \in \mathbb{C}[[x]]$ with $f(0) = 1$, then Equation (4.2) has a solution $g(x) \in \mathbb{C}[[x]]$ with $g(0) = 1$, where*

$$H(x) = \prod_{\alpha_i \in S} (1 - \alpha_i x)^{n_i}$$

is the \mathbb{N}' -cyclotomic part of $G(x)$.

Proof. By Theorem 4.2, it suffices to prove

$$(4.25) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1}\dots\theta_{i_k}|\lambda} \frac{\nu_{i_1}\dots\nu_{i_k}}{e} \frac{\theta_{i_1}\dots\theta_{i_k}}{b} \\ \times \sum_{\substack{\alpha_i \in S \\ \lambda b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1} b_0\dots b_m = \beta}} n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1}} = 0,$$

for all $\lambda \in \mathbb{Q}'_b$, $\lambda \notin \mathbb{N}'$; $u \in \mathbb{N}$, $(u, b_0\dots b_m) = 1$; and $\beta \in \mathbb{C}^*$.

Since all the elements of S are n -th roots of unity, for some $n \in \mathbb{N}'$, Equation (4.25) trivially holds when $\beta^n \neq 1$, for all $n \in \mathbb{N}'$.

Otherwise, assume $\beta^n = 1$ for some $n \in \mathbb{N}'$. In Equation (4.25), the conditions

$$b^{-1}\theta_{i_1}\dots\theta_{i_k} \mid \lambda \text{ and } \lambda \in \mathbb{Q}'_b \Rightarrow \lambda b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1} b_0\dots b_m \in \mathbb{N}'.$$

So if

$$\alpha_i^{\lambda b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1} b_0\dots b_m} = \beta,$$

then α_i is an n -th root of unity for some $n \in \mathbb{N}'$, hence $\alpha_i \in S$.

Sp we can drop the subscription $\alpha_i \in S$ in the summation of (4.25):

$$(4.26) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1}\dots\theta_{i_k}|\lambda} \frac{\nu_{i_1}\dots\nu_{i_k}}{e} \frac{\theta_{i_1}\dots\theta_{i_k}}{b} \\ \times \sum_{\substack{\alpha_i \\ \lambda b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1} b_0\dots b_m = \beta}} n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1}} = 0,$$

Then (4.26) holds by Theorem 4.2 and the assumption $f \in \mathbb{C}[[x]]$. \square

Now assume

$$(4.27) \quad H(x) = \prod_{\alpha_i \in S} (1 - \alpha_i x)^{n_i} = \prod_{d \in \mathbb{N}'} (1 - x^d)^{m_d},$$

where $m_d = 0$ for d sufficiently large. Instead of Theorem 4.2, we have the following simple criterion.

For convenience, we always denote

$$(4.28) \quad m_d = 0 \text{ for } d \notin \mathbb{N}'.$$

Theorem 4.4. *Let*

$$H(x) = \prod_{d \in \mathbb{N}'} (1 - x^d)^{m_d}, \text{ where } m_d = 0 \text{ for } d \gg 0.$$

Then Equation (4.2) has a solution $g(x) \in \mathbb{C}[[x]]$ with $g(0) = 1$ if and only if for any $\lambda \in \mathbb{Q}'_b - \mathbb{N}'$, the following equation holds

$$(4.29) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \nu_{i_1} \dots \nu_{i_k} m_{\lambda \theta_{i_1}^{-1} \dots \theta_{i_k}^{-1}} = 0.$$

Proof. From the proof of Theorem 4.2 (see Equation (4.21)), $g(x) \in \mathbb{C}[[x]]$ if and only if the following equation

$$(4.30) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1}\dots\theta_{i_k}|\lambda} \frac{\nu_{i_1}\dots\nu_{i_k}}{e} \frac{\theta_{i_1}\dots\theta_{i_k}}{b} \sum_{\alpha_i \in S} n_i \alpha_i^{\lambda u b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1}} = 0$$

holds for any $\lambda \in \mathbb{Q}'_b - \mathbb{N}'$ and $u \in \mathbb{N}$, s.t. $(u, b_0 \dots b_m) = 1$.

Computing $\frac{xH'(x)}{H(x)}$ by two expressions of $H(x)$ in (4.27), we get

$$\begin{aligned} \sum_{\alpha_i \in S} \frac{-n_i \alpha_i x}{1 - \alpha_i x} &= \sum_{d \in \mathbb{N}'} \frac{-m_d dx^d}{1 - x^d} \\ - \sum_{m=1}^{+\infty} \left(\sum_{\alpha_i \in S} n_i \alpha_i^m \right) x^m &= - \sum_{m=1}^{+\infty} \left(\sum_{\substack{d|m \\ d \in \mathbb{N}'}} dm_d \right) x^m. \end{aligned}$$

So

$$(4.31) \quad \sum_{\alpha_i \in S} n_i \alpha_i^m = \sum_{d|m, d \in \mathbb{N}'} dm_d = \sum_{d|m} dm_d$$

for any $m \in \mathbb{N}$. The last equality of (4.31) holds because of Equation (4.28).

Substituting (4.31) into (4.30), we get

$$(4.32) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1}\dots\theta_{i_k}|\lambda} \frac{\nu_{i_1}\dots\nu_{i_k}}{e} \frac{\theta_{i_1}\dots\theta_{i_k}}{b} \sum_{d|\lambda u b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1}} dm_d = 0.$$

We can omit u from Equation (4.32):

$$(4.33) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{b^{-1}\theta_{i_1}\dots\theta_{i_k}|\lambda} \frac{\nu_{i_1}\dots\nu_{i_k}}{e} \frac{\theta_{i_1}\dots\theta_{i_k}}{b} \sum_{d|\lambda b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1}} dm_d = 0$$

since $d \in \mathbb{N}'$ (otherwise, $m_d = 0$), $\lambda \in \mathbb{Q}'_b$ and $(u, b_0 \dots b_m) = 1$.

Equation (4.33) is equivalent to

$$(4.34) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \frac{\nu_{i_1}\dots\nu_{i_k}}{e} \frac{\theta_{i_1}\dots\theta_{i_k}}{b} \sum_{d|\lambda b \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1}} dm_d = 0,$$

since

$$d \mid \lambda b \theta_{i_1}^{-1} \dots \theta_{i_k}^{-1} \text{ and } d \in \mathbb{N}' \Rightarrow b^{-1} \theta_{i_1} \dots \theta_{i_k} \mid \lambda.$$

From (4.28), replacing d by $db \theta_{i_1}^{-1} \dots \theta_{i_k}^{-1}$ in Equation (4.34), we get

$$(4.35) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \frac{\nu_{i_1}\dots\nu_{i_k}}{e} \sum_{d|\lambda} dm_{db \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1}} = 0.$$

for any $\lambda \in \mathbb{Q}'_b - \mathbb{N}'$.

In Equation (4.35), since

$$m_{db \theta_{i_1}^{-1}\dots\theta_{i_k}^{-1}} \neq 0 \Rightarrow db \theta_{i_1}^{-1} \dots \theta_{i_k}^{-1} \in \mathbb{N}',$$

we get $d \in \mathbb{Q}'_b$. Also, the conditions:

$$d \mid \lambda \text{ and } \lambda \in \mathbb{Q}'_b - \mathbb{N}' \Rightarrow d \notin \mathbb{N}'.$$

Therefore, $d \in \mathbb{Q}'_b - \mathbb{N}'$, in Equation (4.35).

Changing the order of summation of (4.35), we have

$$(4.36) \quad \sum_{\substack{d \mid \lambda, \\ d \in \mathbb{Q}'_b - \mathbb{N}'}} \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \nu_{i_1} \cdots \nu_{i_k} m_{db\theta_{i_1}^{-1} \dots \theta_{i_k}^{-1}} = 0$$

for any $\lambda \in \mathbb{Q}'_b - \mathbb{N}'$.

By the following modified version of Möbius inversion formula (Lemma 4.5), Equation (4.36) is equivalent to

$$(4.37) \quad \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \nu_{i_1} \cdots \nu_{i_k} m_{db\theta_{i_1}^{-1} \dots \theta_{i_k}^{-1}} = 0$$

for any $d \in \mathbb{Q}'_b - \mathbb{N}'$.

Changing variable d by λ in Equation (4.37), we get the formula (4.29). \square

Lemma 4.5. (*Modified Möbius Inversion formula*) Let $\{A_n\} \in \mathbb{C}$ be a sequence indexed by $n \in \mathbb{Q}'_b - \mathbb{N}'$. For any $m \in \mathbb{Q}'_b - \mathbb{N}'$, define

$$(4.38) \quad B_m = \sum_{\substack{n \mid m \\ n \in \mathbb{Q}'_b - \mathbb{N}'}} A_n.$$

We always assume (4.38) is a finite sum, i.e., there are only finitely many nonzero terms in the summation. Then

$$(4.39) \quad A_n = \sum_{\substack{m \mid n \\ m \in \mathbb{Q}'_b - \mathbb{N}'}} \mu\left(\frac{n}{m}\right) B_m,$$

where μ is the Möbius function.

Proof.

$$\begin{aligned} \sum_{\substack{m \mid n \\ m \in \mathbb{Q}'_b - \mathbb{N}'}} \mu\left(\frac{n}{m}\right) B_m &= \sum_{\substack{m \mid n \\ m \in \mathbb{Q}'_b - \mathbb{N}'}} \mu\left(\frac{n}{m}\right) \sum_{\substack{l \mid m, \\ l \in \mathbb{Q}'_b - \mathbb{N}'}} A_l \\ &= \sum_{\substack{l \mid n \\ l \in \mathbb{Q}'_b - \mathbb{N}'}} A_l \sum_{\substack{l \mid m \mid n \\ m \in \mathbb{Q}'_b - \mathbb{N}'}} \mu\left(\frac{n}{m}\right) \\ &= \sum_{\substack{l \mid n \\ l \in \mathbb{Q}'_b - \mathbb{N}'}} A_l \sum_{\substack{\frac{n}{m} \mid \frac{n}{l} \\ \frac{n}{m} \in \mathbb{N}}} \mu\left(\frac{n}{m}\right) \\ &= A_n. \end{aligned}$$

The second equality from the bottom is because:

$$n, l \in \mathbb{Q}'_b - \mathbb{N}' \Rightarrow \frac{n}{l} \in \mathbb{N}' \Rightarrow \frac{m}{l} \in \mathbb{N}' \Rightarrow m \in \mathbb{Q}'_b - \mathbb{N}'.$$

\square

Theorem 4.6. *Let*

$$H(x) = \prod_{d \in \mathbb{N}'} (1 - x^d)^{m_d}, \text{ where } m_d = 0 \text{ for } d \gg 0.$$

Assume Equation (4.2) has a solution $g(x) \in \mathbb{C}[[x]]$ with $g(0) = 1$. Then

$$(4.40) \quad g(x) = \prod_{d \in \mathbb{N}'} (1 - x^d)^{g_d},$$

where

$$(4.41) \quad g_d = \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \frac{1}{e} \nu_{i_1} \cdots \nu_{i_k} m_{bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}}.$$

Proof. First assume that Equation (4.40) holds. Then

$$(4.42) \quad \begin{aligned} & g(x^{b_0})^{e_0} g(x^{b_1})^{e_1} \cdots g(x^{b_m})^{e_m} \\ &= \prod_{d \in \mathbb{N}'} (1 - x^{b_0 d})^{e_0 g_d} \prod_{d \in \mathbb{N}'} (1 - x^{b_1 d})^{e_1 g_d} \cdots \prod_{d \in \mathbb{N}'} (1 - x^{b_m d})^{e_m g_d} \\ &= \prod_{d \in \mathbb{N}'} (1 - x^d)^{e_0 g_d/b_0 + e_1 g_d/b_1 + \cdots + e_m g_d/b_m}, \end{aligned}$$

where we make the convention:

$$(4.43) \quad g_d = 0 \text{ if } d \notin \mathbb{N}'.$$

From (4.42), Equation (4.2) is equivalent to

$$(4.44) \quad \prod_{d \in \mathbb{N}'} (1 - x^d)^{e_0 g_d/b_0 + e_1 g_d/b_1 + \cdots + e_m g_d/b_m} = \prod_{d \in \mathbb{N}'} (1 - x^d)^{g_d}.$$

By Proposition 3.20, Equation (4.44) is equivalent to

$$(4.45) \quad e_0 g_d/b_0 + e_1 g_d/b_1 + \cdots + e_m g_d/b_m = m_d.$$

Since for $d \in \mathbb{Q}' - \mathbb{N}'$, $g_d = m_d = 0$, Equation (4.45) holds for all $d \in \mathbb{Q}'$.

Changing variable d by bd and multiplying $\frac{1}{e}$ in (4.45), we get

$$(4.46) \quad g_d + \nu_1 g_{d\theta_1^{-1}} + \cdots + \nu_m g_{d\theta_m^{-1}} = \frac{1}{e} m_{bd},$$

where $d \in \mathbb{Q}'$.

Now we will solve Equations (4.43) and (4.46) simultaneously.

Iterating (4.46), we get

$$(4.47) \quad g_d = \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \frac{1}{e} \nu_{i_1} \cdots \nu_{i_k} m_{bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}}$$

for all $d \in \mathbb{Q}'$. Note that (4.47) is actually a finite sum.

Substituting (4.47) into (4.46), then

$$\begin{aligned} & \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \frac{1}{e} \nu_{i_1} \cdots \nu_{i_k} m_{bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}} \\ & + \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_{k+1} \leq m} \frac{1}{e} \nu_{i_1} \cdots \nu_{i_k} \nu_{i_{k+1}} m_{bd\theta_{i_1}^{-1} \cdots \theta_{i_{k+1}}^{-1}} \\ & = \frac{1}{e} m_{bd}. \end{aligned}$$

So (4.47) is really a solution of (4.46).

Now we check the solutions (4.47) also satisfy (4.43).

Assume $d \notin \mathbb{N}'$. We divide it into two cases.

case 1: $d \in \mathbb{Q}'_b - \mathbb{N}'$. From Theorem 4.4, $g_d = 0$.

case 2: $d \notin \mathbb{Q}'_b$. Then

$$m_{bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}} = 0 \text{ since } bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} = \frac{b^{k+1}d}{b_{i_1} \cdots b_{i_k}} \notin \mathbb{N}.$$

By Equation (4.47), $g_d = 0$, too.

Hence, $g_d = 0$ if $d \notin \mathbb{N}'$, which concludes the proof. \square

Let p be a prime. For $a \in \mathbb{Z}, a \neq 0$, let $\text{ord}_p(a)$ be the highest exponent v such that p^v divides a . For $b \in \mathbb{Z}, b \neq 0$, define $\text{ord}_p(a/b) = \text{ord}_p(a) - \text{ord}_p(b)$.

Theorem 4.7. *Let*

$$H(x) = \prod_{d \in \mathbb{N}'} (1 - x^d)^{m_d}, \text{ with } m_d = 0 \text{ for } d \gg 0.$$

Assume there exists a prime p with $\text{ord}_p(b_0) > \text{ord}_p(b_i)$, for $1 \leq i \leq m$. If Equation (4.2) has a solution $g(x) \in \mathbb{C}[[x]]$ with $g(0) = 1$, then

$$g(x) = \prod_{d \in \mathbb{N}'} (1 - x^d)^{g_d}$$

and $g_d = 0$ for $d \gg 0$.

Proof. By Theorem 4.6, it suffices to prove

$$(4.48) \quad g_d = \sum_{k=0}^{+\infty} (-1)^k \sum_{1 \leq i_1, \dots, i_k \leq m} \frac{1}{e} \nu_{i_1} \cdots \nu_{i_k} m_{bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}}$$

is zero for d sufficiently large.

For $1 \leq i \leq m$, let

$$(4.49) \quad \theta_i = \frac{b_i}{b_0} = \rho_i p^{-w_i} \text{ with } \text{ord}_p(\rho_i) = 0 \text{ and } w_i \geq 1.$$

Obviously, $\rho_i > 1$. Let $\rho = \max\{\rho_1, \dots, \rho_m\} > 1$.

Below, we will show that, for sufficiently large d , if $bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} \in \mathbb{N}$, then it is also large. As $m_d = 0$ for $d \gg 0$, this will imply

$$m_{bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}} = 0, \text{ for } d \gg 0.$$

Now assume

$$(4.50) \quad bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} = bd\rho_{i_1}^{-1} \cdots \rho_{i_k}^{-1} p^{w_1} \cdots p^{w_k} \in \mathbb{N}.$$

By (4.49), the fractional part of (4.50), $\rho_{i_1}^{-1} \cdots \rho_{i_k}^{-1}$, has denominator which is not divided by p , so

$$(4.51) \quad bd\rho_{i_1}^{-1} \cdots \rho_{i_k}^{-1} \in \mathbb{N}$$

Then we divide the proof into two cases.

case 1: $k \leq \frac{1}{2} \log_\rho bd$. Then

$$(4.52) \quad bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} \geq bd\rho_{i_1}^{-1} \cdots \rho_{i_k}^{-1} \geq bd\rho^{-k} \geq \sqrt{bd}.$$

case 2: $k > \frac{1}{2} \log_\rho bd$. Then, from Equation (4.51),

$$(4.53) \quad bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} \geq p^{w_{i_1}} \cdots p^{w_{i_k}} \geq p^k \geq p^{\frac{1}{2} \log_\rho bd}.$$

Combining (4.52) and (4.53), we get

$$(4.54) \quad bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} \geq \min\{\sqrt{bd}, p^{\frac{1}{2} \log_\rho bd}\}$$

if $bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1} \in \mathbb{N}$.

As $m_d = 0$ if either $d \notin \mathbb{N}$ or $d \gg 0$, from (4.54), we get

$$m_{bd\theta_{i_1}^{-1} \cdots \theta_{i_k}^{-1}} = 0, \text{ for all } 1 \leq i_1, \dots, i_k \leq m, \text{ if } d \gg 0.$$

Then, from (4.48), we get $g_d = 0$ for $d \gg 0$. □

5. CONTRADICTION

The purpose of this section is to prove the following theorem.

Theorem 5.1. *Assume*

$$H(x) = \prod_{d \in \mathbb{N}'} \Phi_d(x)^{c_d} \text{ with } c_d \in \mathbb{Q} \text{ s.t. } c_1 = -1, c_d = 0 \text{ for } d \gg 0,$$

where $\Phi_d(x)$ be the the cyclotomic polynomial of order d , defined by Equation (5.1) below. Also let $\gcd(b_0, \dots, b_n) = 1$. Then Equation (4.2) has no solution $g(x)$ such that

$$g(x) = \prod_{d \in \mathbb{N}'} (1 - x^d)^{g_d}$$

with $g_d \in \mathbb{Q}$ and $g_d = 0$ for $d \gg 0$.

The conclusion of Theorem 5.1 contradicts to that of Theorem 4.7 under common conditions. The proof of Theorem 5.1 makes use of cyclotomic polynomials and Gauss's lemma. Note that cyclotomic polynomials first appear in the work of Cilleruelo and Rué [2].

We call

$$(5.1) \quad \Phi_n(x) = \prod_{u \in (\mathbb{Z}/n\mathbb{Z})^*} (1 - \exp(u \frac{2\pi i}{n})x)$$

the cyclotomic polynomial of order n , where $(\mathbb{Z}/n\mathbb{Z})^*$ denotes the set of invertible classes modulo n , that is,

$$(\mathbb{Z}/n\mathbb{Z})^* = \{u \in \mathbb{N} \mid 1 \leq u \leq n, (u, n) = 1\}.$$

Note our setting is a little different from the traditional case, in which

$$\Phi_n(x) = \prod_{u \in (\mathbb{Z}/n\mathbb{Z})^*} (x - \exp(u \frac{2\pi i}{n})).$$

However, they differ up to multiplying by ± 1 . The remarkable point is that $\Phi_n(0) = 1$ in our setting.

The following facts about cyclotomic polynomials are well known.

(1) $\Phi_n(x)$ is irreducible in $\mathbb{Z}[x]$. As a consequence, if a polynomial $P(x) \in \mathbb{Z}[x]$ vanishes at a primitive root of unity of order n , then there exists a positive integer s such that $P(x) = \Phi_n(x)^s Q(x)$, where $Q(x) \in \mathbb{Z}[x]$ and $Q(\xi) \neq 0$ for all ξ , n -th primitive roots of unity.

(2) $\{\Phi_n(x) \mid n \in \mathbb{N}\}$ and $\{1 - x^n \mid n \in \mathbb{N}\}$ can represent each other:

$$(5.2) \quad 1 - x^n = \prod_{d|n} \Phi_d(x), \quad \Phi_n(x) = \prod_{d|n} (1 - x^d)^{\mu(\frac{n}{d})}$$

where $\mu(\cdot)$ is the Möbius function. This also implies that

$$\deg \Phi_n(x) = n \prod_{p|n} (1 - \frac{1}{p}) = \varphi(n),$$

where φ is the Euler function.

To Theorem 5.1, we need the following lemma.

Lemma 5.2.

$$(5.3) \quad \Phi_d(x^a) = \prod_{d \langle a|d \rangle | f | ad} \Phi_f(x),$$

where, for $a, d \in \mathbb{N}$,

$$(5.4) \quad \langle a|d \rangle = \prod_{p|(a,d)} p^{\text{ord}_p(a)}.$$

Proof.

$$\begin{aligned} \Phi_d(x^a) &= \prod_{u \in (\mathbb{Z}/d\mathbb{Z})^*} (1 - \exp(u \frac{2\pi i}{d}) x^a) \\ (5.5) \quad &= \prod_{u \in (\mathbb{Z}/d\mathbb{Z})^*} \prod_{1 \leq k \leq a} (1 - \exp(u \frac{2\pi i}{ad}) \exp(k \frac{2\pi i}{a}) x) \\ &= \prod_{u \in (\mathbb{Z}/d\mathbb{Z})^*} \prod_{1 \leq k \leq a} (1 - \exp(\frac{u + kd}{ad} 2\pi i) x) \end{aligned}$$

Assume

$$\xi = \exp(\frac{u + kd}{ad} 2\pi i)$$

is a primitive f -th root of unity. Then f is the smallest positive integer such that $\xi^f = 1$, i.e., $ad | f(u + kd)$.

Obviously, $f \mid ad$. Since $(u, d) = 1$, we have $(u + kd, d) = 1$. Then Equation (5.4) implies that $(u + kd, \langle a|d \rangle d) = 1$. Since $\langle a|d \rangle d \mid f(u + kd)$, we get $\langle a|d \rangle d \mid f$. Thus $\langle a|d \rangle d \mid f \mid ad$.

Since each factor

$$(1 - \exp(\frac{u + kd}{ad} 2\pi i)x)$$

in Equation (5.5) appears one time, we get

$$(5.6) \quad \Phi_d(x^a) \mid \prod_{d \mid \langle a|d \rangle d \mid f \mid ad} \Phi_f(x).$$

The degree of the left hand side of (5.6) is $a\varphi(d)$. The degree of the right hand side is

$$\begin{aligned} \sum_{d \mid \langle a|d \rangle d \mid f \mid ad} \varphi(f) &= \sum_{f' \mid \frac{a}{\langle a|d \rangle}} \varphi(f'd \langle a|d \rangle) \\ &= \varphi(d \langle a|d \rangle) \sum_{f' \mid \frac{a}{\langle a|d \rangle}} \varphi(f') \quad \text{as } (\frac{a}{\langle a|d \rangle}, d \langle a|d \rangle) = 1 \\ &= \varphi(d \langle a|d \rangle) \frac{a}{\langle a|d \rangle} \\ &= \varphi(d)a \quad \text{as } p \mid \langle a|d \rangle \Rightarrow p \mid d. \end{aligned}$$

Therefore,

$$\deg \Phi_d(x^a) = \deg \prod_{d \mid \langle a|d \rangle d \mid f \mid ad} \Phi_f(x).$$

Since their constant terms both equal to 1, they must be equal. \square

Now assume

$$(5.7) \quad g(x) = \prod_{d \in \mathbb{N}} \Phi_d(x)^{h_d}, \text{ with } h_d = 0 \text{ for } d \gg 0.$$

Let $a \in \mathbb{N}$, by Lemma 5.2,

$$\begin{aligned} g(x^a) &= \prod_{d \in \mathbb{N}} \Phi_d(x^a)^{h_d} \\ (5.8) \quad &= \prod_{d \in \mathbb{N}} \prod_{\langle a|d \rangle d \mid f \mid ad} \Phi_f(x)^{h_d} \\ &= \prod_{f \in \mathbb{N}} \Phi_f(x)^{\sum_{\langle a|d \rangle d \mid f \mid ad} h_d}. \end{aligned}$$

For fixed f ,

$$(5.9) \quad \langle a|d \rangle d \mid f \mid ad \Leftrightarrow \langle a|d \rangle a^{-1}d \mid fa^{-1} \mid d.$$

Since $a/\langle a|d \rangle$ and d have no common divisor, we have

$$(5.10) \quad \text{ord}_p(fa^{-1}) \begin{cases} = \text{ord}_p(d), & \text{if } p \mid d, \\ \leq 0, & \text{if } p \nmid d, \end{cases}$$

where p is any prime.

For positive $y \in \mathbb{Q}$, denote

$$(5.11) \quad [y] = \prod_{\text{ord}_p(y) > 0} p^{\text{ord}_p(y)}.$$

From (5.9) and (5.10), we have, for fixed f ,

$$(5.12) \quad \langle a|d \rangle d \mid f \mid ad \Rightarrow d = \left[\frac{f}{a} \right].$$

Combining (5.8) and (5.12), we get

$$(5.13) \quad g(x^a) = \prod_{f \in \mathbb{N}} \Phi_f(x)^{h_{[f/a]}}.$$

From Equation (5.13), we get the following formula.

Lemma 5.3. *Assume*

$$g(x) = \prod_{d \in \mathbb{N}} \Phi_d(x)^{h_d}, \text{ with } h_d = 0 \text{ for } d \gg 0.$$

Then

$$(5.14) \quad g(x^{b_0})^{e_0} g(x^{b_1})^{e_1} \dots g(x^{b_m})^{e_m} = \prod_{d \in \mathbb{N}} \Phi_d(x)^{\sum_i e_i h_{[d/b_i]}}.$$

To prove the main result of this section, we also need Gauss's Lemma. Now we recall it.

Let $p(x) = a_0 + a_1x + \dots + a_nx^n$ be a non-zero polynomial in $\mathbb{Z}[x]$. If the greatest common divisor of a_0, a_1, \dots, a_n is 1, then $p(x)$ is called a primitive polynomial.

Every non-zero polynomial $q(x) \in \mathbb{Q}[x]$ can be uniquely written as

$$q(x) = cq_1(x)$$

with $c > 0$ and $q_1(x) \in \mathbb{Z}[x]$ being primitive. We call c the content of $q(x)$ and denote it by $\text{cont}(q)$. The following version of Gauss's Lemma will be found in page 181 of Lang [5], Theorem 2.1 of Chapter IV.

Theorem 5.4. (*Gauss's Lemma*) *Let $p, q \in \mathbb{Q}[x]$ be non-zero polynomials. Then*

$$\text{cont}(p \cdot q) = \text{cont}(p) \cdot \text{cont}(q).$$

Finally, we can prove Theorem 5.1.

Proof. Assume Equation (4.2) has a solution $g(x)$ such that

$$(5.15) \quad g(x) = \prod_{d \in \mathbb{N}'} (1 - x^d)^{g_d}, \text{ with } g_d \in \mathbb{Q} \text{ and } g_d = 0 \text{ for } d \gg 0.$$

From (5.2), we have

$$g(x) = \prod_{d \in \mathbb{N}'} \Phi_d(x)^{h_d} \text{ with } h_d \in \mathbb{Q} \text{ and } h_d = 0 \text{ for } d \gg 0.$$

From Lemma 5.3, we get

$$(5.16) \quad \prod_{d \in \mathbb{N}'} \Phi_d(x)^{\sum_{i=0}^m e_i h_{[d/b_i]}} = \prod_{d \in \mathbb{N}'} \Phi_d(x)^{c_d}.$$

Since $\Phi_d(x)$ is irreducible in $\mathbb{Z}[x]$, taking some power of Equation (5.16) if necessary, we get the following equations by the uniqueness factorization property of $\mathbb{Z}[x]$:

$$(5.17) \quad \sum_{i=0}^m e_i h_{[d/b_i]} = c_d \text{ for all } d \in \mathbb{N}'.$$

Since $\gcd(b_0, \dots, b_m) = 1$, there exists a prime p such that $p \mid b_0$ but $p \nmid b_i$ for some $1 \leq i \leq m$. Taking $d = p^n$ ($n \geq 0$) in (5.17), we get the following equations

$$(5.18) \quad a_0 h_{[p^n]} + a_1 h_{[p^{n-1}]} + \dots + a_t h_{[p^{n-t}]} = c_{p^n} \quad (n \geq 0)$$

where

$$(5.19) \quad t \geq 1 \text{ and } a_0 a_t \neq 0.$$

To simplify the notations, let $h_{p^n} = H'_n$ and $c_{p^n} = C'_n$. Then Equation (5.18) can be written explicitly as

$$(5.20) \quad \begin{cases} a_0 H'_0 & + a_1 H'_0 & + \dots + a_t H'_0 & = C'_0 = -1 \\ a_0 H'_1 & + a_1 H'_0 & + \dots + a_t H'_0 & = C'_1 \\ a_0 H'_2 & + a_1 H'_1 & + \dots + a_t H'_0 & = C'_2 \\ & \dots \dots & & \\ a_0 H'_t & + a_1 H'_{t-1} & + \dots + a_t H'_0 & = C'_t \\ & \dots \dots & & \\ a_0 H'_{l+1} & + a_1 H'_l & + \dots + a_t H'_{l+1-t} & = C'_{l+1}. \\ & \dots \dots & & \end{cases}$$

So

$$(5.21) \quad H'_0 = -\frac{1}{A}, \text{ where } A = \sum_{i=0}^t a_i = \sum_{i=0}^m e_i.$$

Subtracting the other equations of (5.20) by the first equation, and letting $H_{i-1} = H'_i - H'_0$, $C_{i-1} = C'_i - C'_0$ ($i \geq 1$), we get

$$(5.22) \quad \begin{cases} a_0 H_0 & = C_0 \\ a_0 H_1 + a_1 H_0 & = C_1 \\ \dots \dots & \\ a_0 H_t + a_1 H_{t-1} + \dots + a_t H_0 & = C_t \\ \dots \dots & \\ a_0 H_l + a_1 H_{l-1} + \dots + a_t H_{l-t} & = C_l. \\ \dots \dots & \end{cases}$$

Note that

$$(5.23) \quad H_k = -H'_0 = \frac{1}{A}, \quad C_l = -C'_0 = 1$$

for k, l sufficiently large.

The Equation (5.22) is equivalent to the following identity in $\mathbb{Q}[[z]]$.

$$(5.24) \quad \begin{aligned} & (a_0 + a_1z + \cdots + a_tz^t)(H_0 + H_1z + \cdots + H_kz^k + \cdots) \\ & = C_0 + C_1z + C_2z^2 + \cdots + C_lz^l + \cdots \end{aligned}$$

Let r (resp. s) be the largest k (resp. l) such that (5.23) does not hold. Substituting (5.23) into (5.24), we get

$$(5.25) \quad \begin{aligned} & (a_0 + a_1z + \cdots + a_tz^t)(H_0 + H_1z + \cdots + H_rz^r + \frac{1}{A} \frac{z^{r+1}}{1-z}) \\ & = C_0 + C_1z + \cdots + C_s z^s + \frac{z^{s+1}}{1-z} \end{aligned}$$

Multiplying both sides of Equation (5.25) by $1-z$, we get

$$(5.26) \quad \begin{aligned} & (a_0 + \cdots + a_tz^t)((H_0 + \cdots + H_rz^r)(1-z) + \frac{z^{r+1}}{A}) \\ & = C_0 + (C_1 - C_0)z + \cdots + (C_s - C_{s-1})z^n + (1 - C_s)z^{s+1}. \end{aligned}$$

The right hand side of (5.26) is a primitive polynomial, since their coefficients sum to 1. Let $d = \gcd(a_0, a_1, \dots, a_t)$. From (5.26) and Gauss's Lemma, the content of

$$(5.27) \quad (H_0 + \cdots + H_rz^r)(1-z) + \frac{z^{r+1}}{A} = \frac{1}{d}.$$

So the following polynomial

$$(5.28) \quad d(H_0 + \cdots + H_rz^r)(1-z) + \frac{dz^{r+1}}{A} \in \mathbb{Z}[z],$$

is primitive.

Evaluating (5.28) at $z = 1$, we get

$$(5.29) \quad \frac{d}{A} = \frac{d}{\sum_{i=0}^t a_i} \in \mathbb{Z}.$$

From (5.19),

$$\sum_{i=0}^t a_i > a_0 \geq d > 0.$$

A contradiction! □

6. PROOF OF THEOREM 2.2

Proof. Since $P(x) \in \mathbb{Z}[x]$, with $P(0) = 1$ and $P(1) \neq 0$, it can be factored uniquely as

$$(6.1) \quad P(x) = \prod_{d \in \mathbb{N}' - \{1\}} \Phi_d(x)^{c_d} R(x) = Q(x) R(x),$$

with $R(x) \in \mathbb{Z}[x]$ and $R(\xi) \neq 0$ if $\xi^n = 1$, for some $n \in \mathbb{N}'$.

Let

$$G(x) = \frac{P(x)}{1-x}.$$

Then $H(x)$, the \mathbb{N}' -th cyclotomic part of $G(x)$, can be written as

$$(6.2) \quad H(x) = \frac{Q(x)}{1-x} = \prod_{d \in \mathbb{N}'} \Phi_d(x)^{c_d} = \prod_{d \in \mathbb{N}'} (1-x^d)^{m_d}$$

with $c_d, m_d \in \mathbb{Z}$ such that $c_1 = -1$ and $c_d = m_d = 0$ for $d \gg 0$.

Assume the equation

$$(6.3) \quad f(x^{b_0})^{e_0} f(x^{b_1})^{e_1} \cdots f(x^{b_m})^{e_m} = G(x)$$

has a solution $f(x) \in \mathbb{C}[[x]]$ with $f(0) = 1$.

By Corollary 4.3, the equation

$$(6.4) \quad g(x^{b_0})^{e_0} g(x^{b_1})^{e_1} \cdots g(x^{b_m})^{e_m} = H(x)$$

has a solution $g(x) \in \mathbb{C}[[x]]$ with $g(0) = 1$.

From Theorem 4.6, we get

$$(6.5) \quad g(x) = \prod_{d \in \mathbb{N}'} (1-x^d)^{g_d}$$

with $g_d \in \mathbb{Q}$.

By the assumption and Theorem 4.7, we have

$$(6.6) \quad g_d = 0 \text{ for } d \gg 0.$$

If $\gcd(b_0, b_1, \dots, b_m) = 1$, from Equation (6.2) and Theorem 5.1, we know (6.6) is impossible. A contradiction!

Let $\gcd(b_0, b_1, \dots, b_m) = d > 1$. Then the left hand side of Equation (6.3) is a power series with indeterminate x^d , but the coefficient of x^n of the right hand side is a nonzero constant for n large enough. So we still get a contradiction.

Summing up, Equation (6.3) has no solution $f(x) \in \mathbb{C}[[x]]$ with $f(0) = 1$, which concludes the proof. \square

7. CONJECTURES AND REMARKS

In this section, we will give a conjectural answer to the question of Sárkozy and Sós in the case that all the coefficients of linear forms are positive.

Let k be an integer greater than 1. For $m \geq 1$, let

$$M = \{(1, 1), (k, 1), \dots, (k^{m-1}, 1), (k^m, 1)\}.$$

Motivated by Ruzsa's example, let

$$\mathcal{A} = \left\{ \sum_{i=0}^{+\infty} \varepsilon_i k^{(m+1)i}, \varepsilon_i \in \{0, 1, \dots, k-1\} \right\}.$$

By the uniqueness of k -adic representation, we get the representation function

$$r_M(n, \mathcal{A}) = \#\{(a_0, a_1, \dots, a_m) \mid a_0 + ka_1 + \dots + k^m a_m = n, a_i \in \mathcal{A}\}$$

is 1 for all $n \geq 0$.

We conjecture that these are the complete answers to Sárkozy and Sós's question.

Conjecture. For $m \geq 1$, let

$$M = \{(b_0, e_0), (b_1, e_1), \dots, (b_m, e_m)\}$$

with $1 \leq b_0 < b_1 < \dots < b_m$. There exists an infinite subset \mathcal{A} such that $r_M(n, \mathcal{A})$ is constant for n large enough only if

$$M = \{(1, 1), (k, 1), \dots, (k^{m-1}, 1), (k^m, 1)\}$$

for some $k > 1$.

Our initial plan is to prove the above conjecture for the case $b_0 \geq 2$. But it is not successful. The only problem happens in Theorem 4.7. If it can be improved, the case of $b_0 = 2$ is done by our rest arguments.

Note that Theorem 4.7 does not hold for general b_0, b_1, \dots, b_m with $b_0 \geq 2$ and general $H(x)$. For example, take \mathcal{A} to be the set in Ruzsa's example, that is,

$$\mathcal{A} = \left\{ \sum_{i=0}^{+\infty} \varepsilon_i 2^{2i}, \varepsilon_i \in \{0, 1\} \right\}.$$

Let $g(x)$ be the generating function of \mathcal{A} . From Page 4, we have

$$g(x)g(x^2) = \frac{1}{1-x} \quad \text{and} \quad g(x) = \prod_{n=1}^{+\infty} \left(\frac{1}{1-x^{2^n}} \right)^{(-1)^n}.$$

Take $(b_0, b_1, b_2, b_3) = (2, 3, 4, 6)$, we get

$$g(x^2)g(x^3)g(x^4)g(x^6) = \frac{1}{1-x^2} \frac{1}{1-x^3}.$$

This shows that Theorem 4.7 is not true in general.

For the case $b_0 = 1$, the equation

$$f(x^{b_0})^{e_0} f(x^{b_1})^{e_1} \dots f(x^{b_m})^{e_m} = \frac{P(x)}{1-x} = G(x)$$

always has a power series solution

$$f(x) = \prod_{k=0}^{+\infty} \left(\prod_{1 \leq i_1, \dots, i_k \leq m} G(x^{b_{i_1} \dots b_{i_k}})^{e^{-1} \nu_{i_1} \dots \nu_{i_k}} \right)^{(-1)^k}.$$

To solve Sárkozy and Sós's question, we need to decide whether all the coefficients of $f(x)$ belong to $\{0, 1\}$.

It seems difficult to treat the coefficients of infinite products. For example, the Ramanujan tau function $\tau : \mathbb{N} \rightarrow \mathbb{Z}$ is defined by the following identity in $\mathbb{C}[[q]]$:

$$q \prod_{n=1}^{+\infty} (1 - q^n)^{24} = \sum_{n=1}^{+\infty} \tau(n) q^n.$$

Lehmer conjectured that $\tau(n) \neq 0$ for all n , an assertion sometimes known as Lehmer's conjecture. Lehmer verified the conjecture for $n < 214928639999$ (See page 22 of [1]). This conjecture is still open now.

The above arguments suggest that the case of $b_0 = 1$ is more difficult.

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